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CONDITION REQUIRED FOR THE STABILITY OF ORBITAL MOTION

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SUMMARY

It is proved that to achieve the stability of the orbital motion the condition $h < 0$ is necessary, h being the energy constant.

* * *

1.- We shall consider the motion of a particle in the gravitational field of an arbitrary body (further called the planet), having an axi-symmetrical structure. By strength of the condition of field conservation there exists the energy integral

$$T = V + h, \tag{1}$$

and the validity of the Laplace formula

$$\frac{d^2R}{dt^2} = U + 4h. \tag{2}$$

Here h is the energy constant; T is the kinetic energy of the mass unit of the particle; V is a mass-flow function; $U = 4V + 2QV$; $R = r^2$, where r is the radius-vector;

$$Q = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$

Utilizing the homogeneity of the mass-flow function of the problem of n bodies, Jacobi derived from the formulas (1) and (2) the necessary condition $h < 0$ of solar system's stability.

* NEOBKHODIMOYE USLOVIYE USTOYCHIVOSTI ORBITAL'NOGO DVIZHENIYA

In our case the mass-flow function

$$V = \varkappa^2 \left\{ \frac{1}{r} - \sum_{n=2}^{\infty} J_n \frac{P_n\left(\frac{z}{r}\right)}{r^{n+1}} \right\} \tag{3}$$

is devoid of homogeneity. Here $\varkappa^2$ is the product of the gravitational constant by the mass of the planet; P_n are the Legendre polynomials; J_n are constants. We took for the unit of length the greatest radius-vector of planet's surface.

According to Lagrange, the motion of a material point is said to be steady, if at $t \geqslant t_o$ the point does emerge from the finite region of space **M**.

If we take for the region **M** a sphere of arbitrary radius, it is well known that the condition

$$h < 0. \tag{4}$$

is sufficient for the stability.

We shall demonstrate that the condition (4) is indispensable if we take for **M** any closed finite region of space D, which would not intersect the planet, and in which

$$U > 0. \tag{5}$$

Assume $h \geqslant 0$, and that the particle shall never emerge from the region D. Then, it follows from formulas (2) and (5) :

$$\frac{d^2R}{dt^2} > U_0 > 0. \tag{6}$$

Integrating, we find

$$R - R_0 > \frac{U_0}{2}(t-t_0)^2 + \frac{dR(t_0)}{dt}(t-t_0), \tag{7}$$

whence $R = r^2 \xrightarrow[t \to \infty]{} \infty$, which is in contradiction with the assumption just made. Thus the requirement of the condition (4) has been demonstrated.

2. - It is quite probable that for planets of the solar system the inequality (5) is fulfilled in the whole outer space. If this is so, then at $h \geqslant 0$ the particle <u>either drifts away to infinity</u> or <u>falls on the planet</u>.*

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* This seems evident. However, it is possible to show that there exist force fields in which the motion along the circle $r = r_o > 1$ is possible for any great <u>h</u>.

It has been possible to demonstrate mathematically the inequality (5) for $r \geqslant 1$, for the series (3) may be divergent at $r < 1$. To demonstrate this, let us note first of all that at $r \geqslant 1$ the series (3) may be differenti ated over x, y, z termwise, for the coefficients J_n of the arbitrary body of revolution [1] satisfy the inequality

$$|J_n| \leqslant \frac{C}{n^{5/2}}, \tag{8}$$

where C is a certain constant. That is why the operator Q may be introduced under the sign of the sum (3) and we may take advantage of the Euler formula for an homogenous function. As a result we shall obtain

$$U = \frac{2\varkappa^2}{r}\left\{1 + \sum_{n=2}^{\infty}(n-1)J_n\frac{P_n\left(\frac{z}{r}\right)}{r^n}\right\}. \tag{9}$$

Assume $\quad j_k = \sup_{n>k}\{|J_n| n^{5/2}\} \quad (j_k < C).$

Since

$$\left|\sum_{n=2}^{\infty}(n-1)J_n\frac{P_n}{r^n}\right| < \frac{|J_2|}{r^3} + j_3\sum_{n=3}^{\infty}\frac{1}{r^n n^{3/2}} <$$
$$< \frac{1}{r^3}\left\{|J_2| + J_3\left[\zeta\left(\frac{3}{2}\right) - 1 - \frac{1}{2^{3/2}}\right]\right\} \quad (r \geqslant 1),$$

the quantity U will be strictly positive at $r \geqslant 1$ if

$$|J_2| + j_3\left[\zeta\left(\frac{3}{2}\right) - 1 - \frac{1}{2^{3/2}}\right] = |J_2| + 1{,}26 j_3 < 1. \tag{10}$$

Here $\zeta(x)$ is a Riemann-zeta-function.

For the Earth [2] $j_3 < J_2 \approx 10^{-3}$. Obviously, the inequality (10) is fulfilled for all the planets of the solar system.

Therefore, the motion of a particle in the gravitational field of an axi-symmetrical planet may, according to Lagrange, take place in the M-region only in the case when the inequality $h < 0$ is satisfied.

We may take for the region M any finite part of space $r \geqslant 1$, which coincides with any finite part relative to the planet of the outer space, with a precision to the difference between the equatorial and polar radii of the planet.

*** THE END ***

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